

# INTEGRATING HOLOMORPHIC $L^1$ -FUNCTIONS

A.-K. HERBIG

ABSTRACT. Let  $\Omega \Subset \mathbb{C}^n$  be a domain with smooth boundary,  $k \in \mathbb{N}$ . It is shown that the integral of a holomorphic function in  $L^1(\Omega)$  may be represented as the integral of this function against a smooth function vanishing to order  $k-1$  on  $b\Omega$ . An application for a smoothing property of the Bergman projection for conjugate holomorphic functions is given.

## 1. INTRODUCTION

Let  $\Omega \Subset \mathbb{C}^n$ ,  $n \geq 1$ , be a bounded domain with smooth boundary. Write  $\mathcal{O}(\Omega)$  for the space of holomorphic functions on  $\Omega$ , and denote by  $\mathcal{C}^\infty(\overline{\Omega})$  the space of functions which are smooth up to the boundary,  $b\Omega$ , of  $\Omega$ . The Bergman projection

$$B : L^2(\Omega) \longrightarrow L^2(\Omega) \cap \mathcal{O}(\Omega)$$

is the orthogonal projection of  $L^2(\Omega)$ , the space of square-integrable functions on  $\Omega$ , onto its closed subspace of holomorphic functions.

In [1], Section 6, Bell constructed for each  $k \in \mathbb{N}$  a differential operator  $\Phi^k$  of order  $k$ , with coefficients in  $\mathcal{C}^\infty(\overline{\Omega})$ , satisfying

- (i)  $B(\Phi^k f) = Bf$  for all  $f \in \mathcal{C}^\infty(\overline{\Omega})$ ,
- (ii)  $\Phi^k f$  vanishes to order  $k-1$  on  $b\Omega$  whenever  $f \in \mathcal{C}^\infty(\overline{\Omega})$ .

Since  $\Omega$  is bounded,  $\mathcal{C}^\infty(\overline{\Omega}) \subset L^2(\Omega) \subset L^1(\Omega)$  holds. Hence it may be deduced that for  $\eta \in \mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$

$$\int_{\Omega} \eta dV = \int_{\Omega} B\eta dV \stackrel{(i)}{=} \int_{\Omega} B(\Phi^k \eta) dV = \int_{\Omega} (\Phi^k \eta) \cdot \overline{B(1)} dV$$

holds. Here, in the last step, the self-adjointness of  $B$  (with respect to the Hermitian  $L^2$ -inner product) was used. Since  $B(1) = 1$  it follows that

$$(1.1) \quad \int_{\Omega} \eta dV = \int_{\Omega} \Phi^k \eta dV \quad \forall \eta \in \mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega).$$

The purpose of this note is to extend the integral identity (1.1) to a wider class of functions while replacing the differential operator  $\Phi^k$  on its right hand side by multiplication with a smooth function which vanishes to order  $k-1$  on  $b\Omega$ .

---

2010 *Mathematics Subject Classification.* 32A25, 32A40.

Research supported by the Austrian Science Fund FWF grant V187N13.

**Theorem 1.2.** *Let  $\Omega \Subset \mathbb{C}^n$ ,  $n \geq 1$ , be a smoothly bounded domain. Let  $\delta \in C^\infty(\overline{\Omega})$  be a function which equals the distance-to-the-boundary function for  $\Omega$  near  $b\Omega$ . Let  $k \in \mathbb{N}$  and  $g \in C^\infty(\overline{\Omega})$  be given. Then there exists a function  $\omega_{k,g} \in C^\infty(\overline{\Omega})$  such that*

$$(1.3) \quad \int_{\Omega} \eta \cdot g \, dV = \int_{\Omega} \delta^k \cdot \omega_{k,g} \cdot \eta \, dV \quad \forall \eta \in L^1(\Omega) \cap \mathcal{O}(\Omega).$$

In the special case when  $\eta \in L^2(\Omega) \cap \mathcal{O}(\Omega)$  and  $g \equiv 1$ , identity (1.3) may be derived directly from (1.1) by an integration by parts argument. E.g., when  $k = 1$ , one first notices that Bell's operator  $\Phi^1$  is of the form  $r\tilde{\Phi}^1$  where  $r \in C^\infty(\overline{\Omega})$  is zero on  $b\Omega$  and  $\tilde{\Phi}^1$  is a first order differential operator with coefficients in  $C^\infty(\overline{\Omega})$ . It then follows from the ellipticity of the  $\bar{\partial}$ -operator on functions that  $\tilde{\Phi}^1\eta$  may be written as a tangential derivative of  $\eta$  (plus a lower order term) so that integrating by parts yields a vanishing boundary term (see Lemma 6.1 in [2] for details for this kind of reasoning). This argument lets one rewrite the right hand side of (1.1) in the shape of the right hand side of (1.3).

Theorem 1.2 allows us to reprove, and slightly improve, a smoothing property of the Bergman projection for conjugate holomorphic functions in  $L^2(\Omega)$  previously proved as Theorem 1.10 in [4]. To state this properly, let us fix some notation first. For given  $\ell \in \mathbb{Z}$  denote by  $H^\ell(\Omega)$  the  $L^2$ -Sobolev space of order  $\ell$ , write  $\|\cdot\|_\ell$  for its norm, see Subsection 3.1 for more details. Furthermore, set  $A^0(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$ , and define  $\overline{A^0(\Omega)}$  to be the space of functions whose complex conjugates belong to  $A^0(\Omega)$ .

**Corollary 1.4.** *Let  $\Omega \Subset \mathbb{C}^n$ ,  $n \geq 1$ , be a smoothly bounded domain. Suppose that for a given pair  $(k_1, k_2) \in \mathbb{N}_0^2$  there exists a constant  $C > 0$  such that*

$$(1.5) \quad \|Bf\|_{k_2} \leq C\|f\|_{k_1} \quad \forall f \in H^{k_2}(\Omega).$$

*Let  $k \in \mathbb{N}$  and  $g \in C^\infty(\overline{\Omega})$  be given. Then there exists a constant  $\tilde{C} > 0$  such that*

$$(1.6) \quad \|B(\mu g)\|_{k_2} \leq \tilde{C}\|\mu\|_{-k} \quad \forall \mu \in \overline{A^0(\Omega)}.$$

Corollary 1.4 has been shown to hold in the case  $k = 0$  in [4], Theorem 1.10. This case indicates that the Bergman projection maps the products of the form  $\mu g$  to particularly “nice” holomorphic  $L^2$ -functions. That (1.6) holds for any  $k \in \mathbb{N}$  seems to hint at  $B$  acting on such a product in a “simple” manner. To be less vague the introduction of some more notation is convenient. Denote by  $\overline{A^{-k}(\Omega)}$  the closure of  $A^0(\Omega)$  with respect to  $\|\cdot\|_{-k}$ . Additionally, write  $\overline{A_g^{-k}(\Omega)}$  for the space consisting of functions which are products of  $g$  and functions in  $\overline{A^{-k}(\Omega)}$ .

**Corollary 1.7.** *Suppose the hypotheses of Corollary 1.4 hold. Then the Bergman projection extends to an  $H^{k_2}(\Omega)$ -bounded operator on  $\overline{A_g^{-k}(\Omega)}$ .*

The article is structured as follows. In Section 2 the proof of Theorem 1.2 is given. The proofs of Corollaries 1.4 and 1.7 are provided in Section 3.

**Acknowledgement.** I would like to thank Jeff McNeal for pointing out to me that Corollary 1.7 holds and for his advice on the exposition of this article.

## 2. PROOF OF THEOREM 1.2

**2.1. Setting.** Before presenting the proof of Theorem 1.2 let us give the basic definitions and notions used in this setting.

Let  $\Omega \Subset \mathbb{C}^n$  be a domain with  $\mathcal{C}^\infty$ -boundary. Denote by  $(z_1, \dots, z_n)$  the standard coordinates of  $\mathbb{C}^n$ , write  $x_{2j-1} = \operatorname{Re}(z_j)$  and  $x_{2j} = \operatorname{Im}(z_j)$  for  $j \in \{1, \dots, n\}$ . Let  $\|\cdot\|$  be the Euclidean norm on  $\mathbb{C}^n$ . The distance-to-the-boundary function,  $d_{b\Omega}(z)$ , for  $\Omega$  is given by  $\inf\{\|z - w\| : w \in b\Omega\}$  and satisfies the properties

- (a) there exists a neighborhood  $U$  of  $b\Omega$  such that  $d_{b\Omega}$  is smooth on  $\overline{\Omega} \cap U$ ,
- (b)  $\|\nabla d_{b\Omega}(z)\| = 1$  for all  $z \in \overline{\Omega} \cap U$ .

For proofs of the facts (a) and (b), see Lemma 1, pg. 382, in [3]. It follows from (a) that

$$(2.1) \quad N := \sum_{j=1}^{2n} \frac{\partial d_{b\Omega}}{\partial x_j} \frac{\partial}{\partial x_j}$$

is a smooth vector field on  $\overline{\Omega} \cap U$ . Moreover, (b) implies that  $N(d_{b\Omega}) = 1$  on  $\overline{\Omega} \cap U$ .

As usual, for an open set  $U \subset \mathbb{C}^n$ , let  $\mathcal{C}_c^\infty(U)$  be the space of functions in  $\mathcal{C}^\infty(\overline{U})$  which are compactly supported in  $U$ . Also,  $L^1(\Omega)$  is the space of measurable functions  $f : \Omega \rightarrow \mathbb{C}$  satisfying

$$\int_{\Omega} |f| dV < \infty,$$

where  $dV$  is the Euclidean volume form.

**2.2. Base case.** The proof of Theorem 1.2 will be done by induction on  $k$ . The base case,  $k = 1$ , is covered by the following lemma.

**Lemma 2.2.** *Let  $\Omega \Subset \mathbb{C}^n$ ,  $n \geq 1$ , be a smoothly bounded domain. Let  $\delta \in \mathcal{C}^\infty(\overline{\Omega})$  be a function which equals  $d_{b\Omega}$  near  $b\Omega$ . For any given  $\gamma \in \mathcal{C}^\infty(\overline{\Omega})$  there exists a function  $\omega_{1,\gamma} \in \mathcal{C}^\infty(\overline{\Omega})$  such that*

$$(2.3) \quad \int_{\Omega} \eta \cdot \gamma dV = \int_{\Omega} \delta \cdot \omega_{1,\gamma} \cdot \eta dV \quad \forall \eta \in L^1(\Omega) \cap \mathcal{O}(\Omega).$$

*Proof of Lemma 2.2.* After possibly shrinking the neighborhood  $U$  of  $b\Omega$  described in subsection 2.1, it may be assumed that  $\delta$  equals  $d_{b\Omega}$  on  $\overline{\Omega} \cap U$ . Now choose a function  $\zeta \in \mathcal{C}_c^\infty(U)$  such that  $\zeta \equiv 1$  in some neighborhood  $U' \Subset U$  of  $b\Omega$ . Then

$$\int_{\Omega} \eta \gamma \, dV = \int_{\Omega} \zeta \eta \gamma \, dV + \underbrace{\int_{\Omega} (1 - \zeta) \eta \gamma \, dV}_{=: I_1}.$$

Note that the term

$$I_1 = \int_{\Omega} \delta \left( \frac{1 - \zeta}{\delta} \cdot \gamma \right) \eta \, dV$$

is of the shape as the claimed right hand side of (2.3) since  $1 - \zeta \equiv 0$  in the neighborhood  $U'$  of  $b\Omega$ . The fact that  $N(\delta) = N(d_{b\Omega}) = 1$  on  $\Omega \cap U$  gives

$$(2.4) \quad \int_{\Omega} \eta \gamma \, dV = \int_{\Omega} N(\delta) \zeta \eta \gamma \, dV + I_1.$$

The intermediate goal is to peel  $N$  off  $\delta$  by an integration by parts argument so that  $\delta$  becomes a multiplicative factor. For that, notice first that the product rule yields

$$\begin{aligned} N(\delta) \zeta \eta \gamma &= N(\delta \zeta \eta \gamma) - \delta N(\zeta \eta \gamma) \\ &\stackrel{(2.1)}{=} \sum_{j=1}^{2n} \frac{\partial}{\partial x_j} (\delta_{x_j} \delta \zeta \eta \gamma) - \delta (\Delta \delta) \zeta \eta \gamma - \delta N(\zeta \eta \gamma). \end{aligned}$$

Integrating over  $\Omega$  gives

$$\begin{aligned} (2.5) \quad \int_{\Omega} N(\delta) \zeta \eta \gamma \, dV &= - \int_{\Omega} \delta N(\zeta \eta \gamma) \, dV - \int_{\Omega} \delta (\Delta \delta) \zeta \eta \gamma \, dV \\ &\quad + \int_{\Omega} d \left( \delta \zeta \eta \gamma \sum_{j=1}^{2n} (-1)^{j+1} \delta_{x_j} \, d\hat{x}_j \right), \end{aligned}$$

where  $d\hat{x}_j = dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_{2n}$ . If  $\eta$  was smooth up to  $b\Omega$ , then an application of Stokes' Theorem and the fact that  $\delta = 0$  on  $b\Omega$  would imply that the last term on the right hand side of (2.4) is zero. However,  $\eta \in L^1(\Omega) \cap \mathcal{O}(\Omega)$ , and hence some extra care has to be taken. Let  $\Omega_\epsilon = \{z \in \Omega : d_{b\Omega}(z) > \epsilon\}$  for  $\epsilon > 0$  small. Since both  $\eta$  and  $\gamma$  are smooth in  $\Omega$ , and therefore on  $\overline{\Omega}_\epsilon$ , Stokes' theorem may be used as follows:

$$\begin{aligned} \int_{\Omega_\epsilon} d \left( \delta \zeta \eta \gamma \sum_{j=1}^{2n} (-1)^{j+1} \delta_{x_j} \, d\hat{x}_j \right) &= \epsilon \int_{b\Omega_\epsilon} \zeta \eta \gamma \sum_{j=1}^{2n} (-1)^{j+1} \delta_{x_j} \, d\hat{x}_j \\ &= \epsilon \int_{\Omega_\epsilon} d \left( \zeta \eta \gamma \sum_{j=1}^{2n} (-1)^{j+1} \delta_{x_j} \, d\hat{x}_j \right). \end{aligned}$$

Therefore,

$$\int_{\Omega_\epsilon} d\left(\delta\zeta\eta\gamma \sum_{j=1}^{2n} (-1)^{j+1} \delta_{x_j} d\hat{x}_j\right) = \underbrace{\epsilon \int_{\Omega_\epsilon} N(\zeta\gamma)\eta dV}_{=:I_2(\epsilon)} + \underbrace{\epsilon \int_{\Omega_\epsilon} \zeta\gamma N(\eta) dV}_{=:I_3(\epsilon)}.$$

Since  $\eta \in L^1(\Omega)$  and both  $\zeta$  and  $\gamma$  are smooth up to  $b\Omega$ , it follows that  $I_2(\epsilon)$  is uniformly bounded in  $\epsilon$ . Therefore  $\epsilon I_2(\epsilon)$  approaches 0 as  $\epsilon$  goes to  $0^+$ . To understand the behavior of the term  $I_3(\epsilon)$ , the holomorphicity of  $\eta$  will be used to transform  $N\eta$  into a tangential derivative of  $\eta$ . For that define

$$T := \sum_{j=1}^n \delta_{x_{2j}} \frac{\partial}{\partial x_{2j-1}} - \delta_{x_{2j-1}} \frac{\partial}{\partial x_{2j}}.$$

Then  $T$  is a smooth vector field on  $\bar{\Omega}$ . Moreover,  $T$  is tangential to the level sets of  $\delta$  as  $T(\delta) \equiv 0$  holds. Additionally, the holomorphicity of  $\eta$  gives

$$N\eta = \sum_{j=1}^n (\delta_{x_{2j-1}} \eta_{x_{2j-1}} + \delta_{x_{2j}} \eta_{x_{2j}}) = i \sum_{j=1}^n (-\delta_{x_{2j-1}} \eta_{x_{2j}} + \delta_{x_{2j}} \eta_{x_{2j-1}}) = iT\eta.$$

Hence

$$(2.6) \quad I_3(\epsilon) = i \int_{\Omega_\epsilon} \zeta\gamma T(\eta) dV = i \int_{\Omega_\epsilon} T(\zeta\gamma\eta) dV - i \int_{\Omega_\epsilon} T(\zeta\gamma)\eta dV.$$

The last term on the right hand side of (2.6) is uniformly bounded in  $\epsilon$  since  $\eta \in L^1(\Omega)$  and  $\zeta, \gamma \in C^\infty(\bar{\Omega})$ . Moreover, the second to last term on the right hand side actually is zero. This follows from yet another application of Stokes' theorem and the facts that  $T$  is tangential to  $b\Omega_\epsilon$ , self-adjoint and  $\eta, \gamma, \zeta \in C^\infty(\Omega)$ . Therefore,  $I_3(\epsilon)$  goes to 0 as  $\epsilon \rightarrow 0^+$ . This concludes the proof of the last term on the right hand side of (2.5) being zero.

Combining the two identities (2.4) and (2.5) then gives

$$\begin{aligned} \int_{\Omega} \eta\gamma dV &= - \int_{\Omega} \delta N(\zeta\eta\gamma) dV - \int_{\Omega} \delta(\Delta\delta)\zeta\eta\gamma dV + I_1 \\ &= - \int_{\Omega} \delta N(\eta)\zeta\gamma dV - \underbrace{\int_{\Omega} \delta\eta (N(\zeta\gamma) + (\Delta\delta)\zeta\gamma) dV}_{=:I_4} + I_1. \end{aligned}$$

Note that the term  $I_4$  is of the shape as the claimed right hand side of (2.3). So the intermediate goal of peeling  $N$  off  $\delta$  has been achieved at the cost of introducing an  $N$ -derivative of  $\eta$ . However, the holomorphicity of  $\eta$  lets us repeat the trick leading up to (2.6) to obtain

$$\begin{aligned} \int_{\Omega} \eta\gamma dV &= -i \int_{\Omega} \delta T(\eta)\zeta\gamma dV + I_1 + I_4 \\ (2.7) \quad &= -i \int_{\Omega} T(\delta\eta\zeta\gamma) dV + i \underbrace{\int_{\Omega} \delta\eta T(\zeta\gamma) dV}_{=:I_5} + I_1 + I_4, \end{aligned}$$

where  $I_5$  is of the right shape with respect to the claimed (2.3). Also, arguments analogously to the ones following (2.6) result in the first term on the right hand side of (2.7) being equal to zero.

Hence it is shown that

$$\int_{\Omega} \eta \gamma dV = I_1 + I_4 + I_5,$$

i.e., (2.3) holds with

$$\omega_{1,\gamma} = \frac{1-\zeta}{\delta} \gamma - (N(\zeta \gamma) + (\triangle \delta) \zeta \gamma - iT(\zeta \gamma)).$$

□

**2.3. Proof of Theorem 1.2.** The proof is done by induction on  $k$ . The case  $k = 1$  is Lemma 2.2 with  $\gamma = g$ . Suppose now that for some  $k \in \mathbb{N}$  there exists a function  $\omega_{k,g} \in C^\infty(\overline{\Omega})$  such that

$$(2.8) \quad \int_{\Omega} \eta \cdot g dV = \int_{\Omega} \delta^k \cdot \omega_{k,g} \cdot \eta dV \quad \forall \eta \in L^1(\Omega) \cap \mathcal{O}(\Omega)$$

holds. Let the neighborhood  $U$  of  $b\Omega$  and the cut-off function  $\zeta$  be given as in the proof of Lemma 2.2 and proceed analogously to the arguments leading up to (2.4). That is,

$$\int_{\Omega} \eta g dV \stackrel{(2.8)}{=} \int_{\Omega} \delta^k \omega_{k,g} \eta dV = \int_{\Omega} \delta^k \zeta \eta \omega_{k,g} dV + \int_{\Omega} \delta^k (1 - \zeta) \omega_{k,g} \eta dV.$$

Moreover, since  $N(\delta) \equiv 1$  on  $\Omega \cap U$ , it follows that

$$\int_{\Omega} \eta g dV = \frac{1}{k+1} \int_{\Omega} N(\delta^{k+1}) \zeta \omega_{k,g} \eta dV + \int_{\Omega} \delta^{k+1} \frac{1-\zeta}{\delta} \omega_{k,g} \eta dV.$$

Now repeat the arguments starting from (2.4) with  $\delta^{k+1}$  in place of  $\delta$  and  $\omega_{k,g}/(k+1)$  in place of  $\gamma$  there to obtain

$$\int_{\Omega} \eta \cdot g dV = \int_{\Omega} \delta^{k+1} \cdot \omega_{k+1,g} \cdot \eta dV$$

with

$$\omega_{k+1,g} = \frac{1-\zeta}{\delta} \omega_{k,g} - \frac{1}{k+1} (N(\zeta \omega_k) + (\triangle \delta) \omega_{k,g} - iT(\zeta \omega_{k,g})).$$

□

### 3. PROOF OF COROLLARIES 1.4 AND 1.7

**3.1. Setting.** Let  $\Omega \Subset \mathbb{C}^n$  be a domain with  $C^\infty$ -boundary. Define, as usual,

$$L^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} : f \text{ measurable, } \int_{\Omega} |f|^2 dV < \infty\}.$$

Write  $(\cdot, \cdot)$  for the usual Hermitian  $L^2$ -inner product, i.e.,

$$(f, g) = \int_{\Omega} f \cdot \overline{g} dV$$

for  $f, g \in L^2(\Omega)$ . Denote by  $\|\cdot\|$  the induced norm. For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{N}_0^{2n}$  of length  $|\alpha| = \sum_{j=1}^{2n} \alpha_j$  define the differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_{2n}^{\alpha_{2n}}}.$$

The  $L^2$ -Sobolev space,  $H^k(\Omega)$ , of order  $k \in \mathbb{N}$  is defined as

$$H^k(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \leq k\};$$

here  $D^\alpha f$  for  $f \in L^2(\Omega)$  is understood in the sense of distributions.  $H^k(\Omega)$  is equipped with the inner product

$$(f, g)_k := \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g) \quad \text{for } f, g \in H^k(\Omega),$$

and is in fact a Hilbert space with the induced norm  $\|\cdot\|_k$ . Set  $A^k(\Omega)$  to be the subspace  $H^k(\Omega) \cap \mathcal{O}(\Omega)$ , and denote by  $\overline{A^k(\Omega)}$  the space of functions whose complex conjugates belong to  $A^k(\Omega)$ .

Let  $H_0^k(\Omega)$  be the closure of  $\mathcal{C}_c^\infty(\Omega)$  with respect to  $\|\cdot\|_k$ . Denote by  $H^{-k}(\Omega)$  the dual space of  $H_0^k(\Omega)$ . The space  $H^{-k}(\Omega)$  is then endowed with the operator norm, i.e., if  $f \in H^{-k}(\Omega)$  then

$$\|f\|_{-k} = \sup \left\{ |(f, g)| : g \in H_0^k(\Omega), \|g\|_k \leq 1 \right\}.$$

For functions in  $A^0(\Omega)$  this norm reduces to an  $L^2$ -norm weighted with the appropriate power of the distance-to-the-boundary function. That is, there exists a constant  $c > 0$  such that

$$(3.1) \quad \frac{1}{c} \|h\|_{-k} \leq \|\delta^k h\| \leq c \|h\|_{-k} \quad \forall h \in A^0(\Omega).$$

Clearly, (3.1) also holds for all  $h \in \overline{A^0(\Omega)}$ . Write  $\overline{A^{-k}(\Omega)}$  for the closure of  $A^0(\Omega)$  with respect to  $\|\cdot\|_{-k}$ .

**3.2. Proof of Corollary 1.4.** Suppose (1.5) holds, i.e., for a given pair  $(k_1, k_2) \in \mathbb{N}_0^2$  there exists a constant  $C > 0$  such that

$$\|Bf\|_{k_2} \leq C \|f\|_{k_1} \quad \forall f \in H^{k_1}(\Omega).$$

Proposition 2.3 in [4] states that under this regularity assumption on  $B$  there exists a constant  $c_1 > 0$  such that

$$(3.2) \quad \|f\|_{k_2} \leq c_1 \sup \left\{ |(f, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \right\}$$

for all  $f \in A^0(\Omega)$ . The case  $k_1 = 0 = k_2$  is not contained in [4], however it is obtained easily from

$$\begin{aligned} \|f\| &= \|Bf\| = \sup \left\{ |(Bf, g)| : g \in L^2(\Omega), \|g\| \leq 1 \right\} \\ &= \sup \left\{ |(f, Bg)| : g \in L^2(\Omega), \|g\| \leq 1 \right\} \end{aligned}$$

and the fact that  $B$  is bounded in  $L^2(\Omega)$ .

Let  $k \in \mathbb{N}$  and  $g \in \mathcal{C}^\infty(\overline{\Omega})$  be given. It then follows from (3.2) that

$$\begin{aligned} \|B(\mu g)\|_{k_2} &\leq c_1 \sup \left\{ |(B(\mu g), h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \right\} \\ &= c_1 \sup \left\{ |(\mu g, h)| : h \in A^{k_2}(\Omega), \|h\|_{-k_1} \leq 1 \right\} \end{aligned}$$

for all  $\mu \in \overline{A^0(\Omega)}$ . Observe first that  $\overline{\mu}h$  is holomorphic in  $\Omega$ , then notice that Hölder's inequality implies that  $\overline{\mu}h \in L^1(\Omega)$ . Hence Theorem 1.2 is applicable here. In particular, for any  $k \in \mathbb{N}$  there exists a function  $\omega_{k+k_1, g} \in \mathcal{C}^\infty(\overline{\Omega})$  such that

$$(\mu g, h) = \left( \delta^k \cdot \omega_{k+k_1, g} \cdot \mu, \delta^{k_1} \cdot h \right).$$

Applying (the right side of) (3.1) twice after using the Cauchy-Schwarz inequality then gives

$$|(\mu g, h)| \leq \|\delta^k \omega_{k+k_1, g} \mu\| \cdot \|\delta^{k_1} h\| \leq c_2 \|\mu\|_{-k} \cdot \|h\|_{-k_1}.$$

Here  $c_2 > 0$  is a constant depending on  $\omega_{k+k_1, g}$  and on the constant  $c$  in (3.1). Thus

$$\|B(\mu g)\|_{k_2} \leq c_1 c_2 \|\mu\|_{-k}.$$

□

**3.3. Proof of Corollary 1.7.** Let  $\mu \in \overline{A^{-k}(\Omega)}$ . Then by definition there exists a sequence  $\{\mu_j\}_j \in \overline{A^0(\Omega)}$  which converges to  $\mu$  in  $\|\cdot\|_{-k}$ . Moreover, it follows from (1.6) that

$$\|B(\mu_j g) - B(\mu_\ell g)\|_{k_2} = \|B((\mu_j - \mu_\ell)g)\|_{k_2} \leq \tilde{C} \|\mu_j - \mu_\ell\|_{-k}.$$

Therefore  $\{B(\mu_j g)\}_j$  is a Cauchy sequence with respect to  $\|\cdot\|_{k_2}$  and hence has a limit in  $H^{k_2}(\Omega)$ . Thus an extension,  $\tilde{B}$ , of the Bergman projection  $B$  may be defined by setting  $\tilde{B}(\mu g)$  to be the limit of  $\{B(\mu_j g)\}_j$  in  $H^{k_2}(\Omega)$ . It then follows that for any  $\epsilon > 0$  there exists a  $j_0 \in \mathbb{N}$  such that for all  $j \geq j_0$

$$\begin{aligned} \|\tilde{B}(\mu g)\|_{k_2} &\leq \|\tilde{B}(\mu g) - B(\mu_j g)\|_{k_2} + \|B(\mu_j g)\|_{k_2} \\ &\leq \epsilon + \tilde{C} \|\mu_j\|_{-k} \\ &\leq \epsilon + \tilde{C} (\|\mu - \mu_j\|_{-k} + \|\mu\|_{-k}) \\ &\leq \epsilon(1 + \tilde{C}) + \tilde{C} \|\mu\|_{-k}. \end{aligned}$$

holds. Therefore,  $\tilde{B}$  is a bounded in  $\|\cdot\|_{k_2}$ . □

## REFERENCES

- [1] Steven R. Bell, Biholomorphic mappings and the  $\bar{\partial}$ -problem, *Ann. of Math. (2)*, 114(1): 103–113, 1981.
- [2] Harold P. Boas, Holomorphic reproducing kernels in Reinhardt domains, *Pacific J. Math.*, 112(2): 273–292, 1984.



- [3] David Gilbarg und Neil Trudinger, *Elliptic partial differential equations of second order, 2nd Ed.*, volume 224 of *Grundlehren der Mathematischen Wissenschaften*, Springer Verlag, New York, 1983.
- [4] A.-K. Herbig, J. D. McNeal, and E. J. Straube, Duality of holomorphic function spaces and smoothing properties of the Bergman projection, to appear in *Transactions of the AMS*.

DEPARTMENT OF MATHEMATICS,  
UNIVERSITY OF VIENNA, VIENNA, AUSTRIA  
*E-mail address:* `anne-katrin.herbig@univie.ac.at`